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Some Coupled Coincidence Point Results Without Compatibility

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ABSTRACT

In this article, without using the notion of compatible mappings, we prove the existence and uniqueness of coupled coincidence points for mixed g-monotone mappings in the setting of partially ordered metric spaces, these results improve and generalize some well-known results in the literature.

Keywords: coupled coincidence point, partially ordered metric space, g-mixed monotone property.

1. Introduction

Existence of coupled fixed points in partially ordered sets has been studied in a number of works. Some recent references on this topic are the works noted in ((Choudhury and Kundu, 2012, Razani and Parvaneh, 2012, Samet and Cojbasic, 2013)).The notions of coupled fixed point and mixed monotone property in metric spaces endowed with a partial order were introduced by Bhaskar and Lakshmikantham (2006)as follows.

Definition 1.1. Let X be a nonempty set and $F : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$, is called a coupled fixed point of F if

$$x = F(x, y) \text{ and } y = F(y, x).$$

Definition 1.2. Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ be a mapping. We say that the mapping F has the mixed monotone property if F is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument. That is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

The notions of coupled coincidence point and mixed g-monotone property have been established recently by Lakshmikantham and Ćirić (2009). These concepts are defined as follows.

Definition 1.3. Let X be a nonempty set and let $F : X \times X \to X$ and $g : X \to X$ be mappings. We say that $(x, y) \in X \times X$ is a coupled coincidence point of F and g if $F(x, y) = g(x)$ and $F(y, x) = g(y)$ for $x, y \in X$.

Definition 1.4. Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$ be mappings. We say F has the mixed g-monotone property if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument. That is, for any $x, y \in X$,

$$x_1, x_2 \in X, g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Recently, the notion of compatibility of F and g was defined by Choudhury and Kundu (2010) as follows.

Definition 1.5. Let (X, d) be a metric space and let $g : X \to X, F : X \times X \to X$. The mappings g and F are compatible if

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0,$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$, for all $x, y \in X$ are satisfied.

Luong and Thuan (2011) have presented some coupled fixed point theorems involving a (ϕ, ψ)-contractive condition.

Let Φ denote all functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy

1. ϕ is continuous and nondecreasing;
2. $\phi(t) = 0$ if and only if $t = 0$;
3. $\phi(t + s) \leq \phi(t) + \phi(s)$, $\forall s, t \in [0, \infty)$;

and let Ψ denote all the functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy $\lim_{t \to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t \to 0^+} \psi(t) = 0$.

For example (Luong and Thuan (2011)), functions $\phi_1(t) = kt$ where $k > 0$, $\phi_2(t) = \frac{t}{t+1}$, $\phi_3(t) = \ln(t+1)$, and $\phi_4(t) = \min\{t, 1\}$ are in Φ; $\psi_1(t) = kt$ where $k > 0$, $\psi_2(t) = \frac{\ln(2t+1)}{2}$, and

$$\psi_3(t) = \begin{cases} 1, & t = 0 \\ \frac{t}{t+1}, & 0 < t < 1 \\ 1, & t = 1 \\ \frac{1}{2}t, & t > 1 \end{cases}$$

are in Ψ.

The following theorem is the main theoretical result of Luong and Thuan (2011) which is generalization of the results of Bhaskar and Lakshmikantham (2006).

Theorem 1.6. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ be a mapping which has the following conditions:*

1. *suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*
$$\phi(d(F(x, y), F(u, v))) \leq \frac{1}{2}\phi(d(x, u) + d(y, v)) - \psi\Big(\frac{d(x, u) + d(y, v)}{2}\Big)$$
for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$;
2. *there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$;*
3. *F has the mixed monotone property;*

4. (a) *F is continuous or*
   (b) *X has the following properties:*
      (i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n;*
      (ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.*

Then F has a coupled fixed point in X.

The purpose of this article is to present some results of coupled coincidence points for g-mixed monotone mappings in partially ordered metric spaces without compatibility which are generalizations of the results of Alotaibi and Alsulami (2011), Bhaskar and Lakshmikantham (2006) and Luong and Thuan (2011).

2. Existence of coupled coincidence points without compatibility

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $g : X \to X$ and $F : X \times X \to X$ mappings have the following conditions:*

1. *$g(X)$ is complete, g is continuous and increasing;*
2. *$F(X \times X) \subset g(X)$;*
3. *assume that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned}\phi(d(F(x,y),F(u,v))) \quad \leq \quad & \frac{1}{2}\phi(d(g(x),g(u))+d(g(y),g(v))) - \\ & \psi\Big(\frac{d(g(x),g(u))+d(g(y),g(v))}{2}\Big)\end{aligned}$$

   for all $x, y, u, v \in X$ with $g(x) \succeq g(u)$ and $g(y) \preceq g(v)$;
4. *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$;*
5. *F has the mixed g-monotone property;*
6. (a) *F is continuous or*
   (b) *X has the following properties:*
      (i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
      (ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.*

Then g and F have a coupled coincidence point.

Proof. Let $U : g(X) \times g(X) \to g(X)$ be a map which is defined by

$$U(g(x), g(y)) = F(x, y) \tag{2.1}$$

for all $g(x), g(y) \in g(X)$. As g is increasing, U is well-defined on $g(X)$. Now, we want to show that $U : g(X) \times g(X) \to g(X)$ satisfies the assumptions of Theorem 1.6 . From condition (3) and (2.1) we have

$$\begin{aligned}\phi(d(U(g(x), g(y)), U(g(u), g(v)))) \quad \leq \quad & \frac{1}{2}\phi(d(g(x), g(u)) + d(g(y), g(v))) - \\ & \psi\Big(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\Big);\end{aligned}$$

for all $g(x), g(y), g(u), g(v) \in g(X)$ with $g(x) \succeq g(u)$ and $g(y) \preceq g(v)$. From condition (4), there exist $g(x_0), g(y_0) \in g(X)$ such that

$$g(x_0) \preceq U(g(x_0), g(y_0)) \text{ and } g(y_0) \succeq U(g(y_0), g(x_0)).$$

From condition (5), if $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$ then $U(g(x), g(y)) \preceq U(g(u), g(v))$. Thus, U has the mixed monotone property. From condition (6) either F is continuous so U is continuous. Or condition (b) holds. Since g is continuous and increasing, $g(X)$ has the following properties:

(i`) if a nondecreasing sequence $\{g(x_n)\} \to g(x)$, then $g(x_n) \preceq g(x)$ for all n,

(ii`) if a nonincreasing sequence $\{g(y_n)\} \to g(y)$, then $g(y_n) \succeq g(y)$ for all n.

Hence, all the assumption of Theorem 1.6 are satisfied. Therefore, U has a coupled fixed point $(x^*, y^*) \in g(X) \times g(X)$. That is, $x^* = U(x^*, y^*)$ and $y^* = U(y^*, x^*)$. Since $x^*, y^* \in g(X)$ then there exist $x, y \in X$ such that $g(x) = x^*, g(y) = y^*$. Thus, $gx = U(gx, gy)$ and $g(y) = U(g(y), g(x))$, i.e., $g(x) = F(x, y)$ and $g(y) = F(y, x)$. Therefore, (x, y) is a coupled coincidence point of F and g. □

If we take $\phi(t) = t$, we get the following corollary.

Corollary 2.2. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $g : X \to X$ and $F : X \times X \to X$ mappings have the following conditions:*

1. *$g(X)$ is complete, g is continuous and increasing;*

2. *$F(X \times X) \subset g(X)$;*
3. *assume that there exists $\psi \in \Psi$ such that*

$$d(F(x,y),F(u,v)) \leq \frac{d(g(x),g(u)) + d(g(y),g(v)}{2} - \psi\Big(\frac{d(g(x),g(u)) + d(g(y),g(v))}{2}\Big)$$

   for all $x,y,u,v \in X$ with $g(x) \succeq g(u)$ and $g(y) \preceq g(v)$;
4. *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$;*
5. *F has the mixed g-monotone property;*
6. *(a) F is continuous or*
   (b) X has the following properties:
   (i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,
   (ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.

Then g and F have a coupled coincidence point.

Let $\psi(t) = (1-k)t$ where $k \in [0,1)$ in corollary 2.2, we obtain the following result.

Corollary 2.3. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $g : X \to X$ and $F : X \times X \to X$ mappings have the following conditions:*

1. *$g(X)$ is complete, g is continuous and increasing;*
2. *$F(X \times X) \subset g(X)$;*
3. *assume that there exists $k \in [0,1)$ such that*

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(g(x),g(u)) + d(g(y),g(v)]$$

   for all $x,y,u,v \in X$ with $g(x) \succeq g(u)$ and $g(y) \preceq g(v)$;
4. *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$;*
5. *F has the mixed g-monotone property;*
6. *(a) F is continuous or*
   (b) X has the following properties:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.

Then g and F have a coupled coincidence point.

Remark 2.4. Theorem 2.1 is extension of the main result of Alotaibi and Alsulami (2011) without using the compatibility. Corollary 2.2 and Corollary 2.3 are generalizations of the results of Luong and Thuan (2011) and Bhaskar and Lakshmikantham (2006), respectively.

When we take $\psi(t) = t$ in Theorem 2.1, we have the following corollary:

Corollary 2.5. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $g : X \to X$ and $F : X \times X \to X$ mappings have the following conditions:*

1. *$g(X)$ is complete, g is continuous and increasing;*
2. *$F(X \times X) \subset g(X)$;*
3. *assume that there exists $\phi \in \Phi$ such that*

$$\phi(d(F(x,y), F(u,v))) \leq \frac{1}{2}\phi(d(g(x), g(u)) + d(g(y), g(v)) - \frac{d(g(x), g(u)) + d(g(y), g(v))}{2}$$

for all $x, y, u, v \in X$ with $g(x) \succeq g(u)$ and $g(y) \preceq g(v)$;

4. *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$;*
5. *F has the mixed g-monotone property;*
6. *(a) F is continuous or*
   (b) X has the following properties:

   (i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,

   (ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.

Then g and F have a coupled coincidence point.

If $\psi(t) = 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.6. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $g : X \to X$ and $F : X \times X \to X$ mappings have the following conditions:*

1. *$g(X)$ is complete, g is continuous and increasing;*
2. *$F(X \times X) \subset g(X)$;*
3. *assume that there exists $\phi \in \Phi$ such that*

$$\phi(d(F(x,y),F(u,v))) \leq \frac{1}{2}\phi(d(g(x),g(u)) + d(g(y),g(v))$$

   for all $x,y,u,v \in X$ with $g(x) \succeq g(u)$ and $g(y) \preceq g(v)$;

4. *there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$;*
5. *F has the mixed g-monotone property;*
6. *(a) F is continuous or*
   (b) X has the following properties:
   - *(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
   - *(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.*

Then g and F have a coupled coincidence point.

If $g = I_X$, I_X is the identity map on X in Corollary 2.2 and Corollary 2.3, then we obtain the following results.

Corollary 2.7. *(Luong and Thuan (2011)) Let $(X, d, \preceq)$ be a partially ordered metric space. Let $F : X \times X \to X$ be a mapping has the following conditions:*

1. *assume that there exists $\psi \in \Psi$ such that*

$$d(F(x,y),F(u,v)) \leq \frac{d(x,u)+d(y,v)}{2} - \psi\Big(\frac{d(x,u)+d(y,v)}{2}\Big)$$

   for all $x,y,u,v \in X$ with $x \succeq u$ and $y \preceq v$;

2. *there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$;*
3. *F has the mixed monotone property;*
4. *(a) F is continuous or*
   (b) X has the following properties:
   - *(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
   - *(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.*

Then F has a coupled fixed point.

Corollary 2.8. *(Bhaskar and Lakshmikantham (2006)) Let $(X, d, \preceq)$ be a partially ordered metric space. Let $F : X \times X \to X$ be a mapping has the following conditions:*

1. *assume that there exists $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \le \frac{k}{2}[d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$;

2. *there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$;*

3. *F has the mixed monotone property;*

4. *(a) F is continuous or*
(b) X has the following properties:

   (i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,

   (ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.

Then F has a coupled fixed point.

3. Uniqueness of coupled coincidence point

In this section, the uniqueness of the coupled coincidence point has been proved. Let $(X, \preceq)$ be a partially ordered set and $(x, y), (u, v) \in X \times X$. We say that $(x, y) \preceq_2 (u, v)$ if $x \preceq u$ and $y \succeq v$.

Theorem 3.1. *In addition to hypotheses of Theorem 2.1, if for any (x, y), $(u, v) \in X \times X$ there exists $(w, z) \in X \times X$ such that $(w, z) \preceq_2 (x, y)$ and $(w, z) \preceq_2 (u, v)$, then (x, y) is the unique coupled coincidence point of F and g.*

Proof. From Theorem 2.1, we have nonempty set of coupled coincidence point of F and g. Let (x, y) and (u, v) be coupled coincidence points of F and g. We want to show that $x = u$ and $y = v$. Suppose that (w, z) is an element of $X \times X$ such that $(w, z) \preceq_2 (x, y)$ and $(w, z) \preceq_2 (u, v)$. Let $(w, z) \preceq_2 (x, y)$. We construct the sequences $g(w_n)$ and $g(z_n)$ as follows:

$$w_0 = w, \quad z_0 = z, \quad g(w_{n+1}) = F(w_n, z_n), \quad g(z_{n+1}) = F(z_n, w_n).$$

Since $(w_0, z_0) \preceq_2 (x, y)$, $w_0 \preceq x$ and $z_0 \succeq y$. g is increasing, thus

$$g(w_0) \preceq g(x) \text{ and } g(z_0) \succeq g(y).$$

As F has g-mixed monotone property, we get

$$F(w_0, z_0) \preceq F(x, y) \text{ and } F(z_0, w_0) \succeq F(y, x).$$

That is,

$$g(w_1) \preceq g(x) \text{ and } g(z_1) \succeq g(y),$$

again by using the property of F, we have

$$F(w_1, z_1) \preceq F(x, y) \text{ and } F(z_1, w_1) \succeq F(y, x).$$

Therefore,

$$g(w_2) \preceq g(x) \text{ and } g(z_2) \succeq g(y).$$

By continuing this process, we obtain that

$$g(w_n) \preceq g(x) \text{ and } g(z_n) \succeq g(y) \text{ for each } n \in \mathbb{N}.$$

When we use the condition (3) of Theorem 2.1, we get

$$\begin{aligned}
\phi(d(g(x), g(w_{n+1}))) &= \phi(d(F(x, y), F(w_n, z_n))) \\
&\leq \frac{1}{2}\phi(d(g(x), g(w_n)) + d(g(y), g(z_n))) - \\
&\quad \psi\Big(\frac{d(g(x), g(w_n)) + d(g(y), g(z_n))}{2}\Big). \qquad (3.1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\phi(d(g(y), g(z_{n+1}))) &= \phi(d(F(y, x), F(z_n, w_n))) \\
&\leq \frac{1}{2}\phi(d(g(x), g(w_n)) + d(g(y), g(z_n))) - \\
&\quad \psi\Big(\frac{d(g(x), g(w_n)) + d(g(y), g(z_n))}{2}\Big). \qquad (3.2)
\end{aligned}$$

By summing (3.1) and (3.2), we deduce

$$\begin{aligned}
\phi(d(g(x), g(w_{n+1}))) + \phi(d(g(y), g(z_{n+1}))) &\leq \phi(d(g(x), g(w_n)) + d(g(y), g(z_n))) - \\
&\quad 2\psi\Big(\frac{d(g(x), g(w_n)) + d(g(y), g(z_n))}{2}\Big).
\end{aligned}$$

That is,

$$\begin{aligned}
\phi(d(g(x), g(w_{n+1})) + d(g(y), g(z_{n+1}))) \leq \phi(d(g(x), g(w_n)) + d(g(y), g(z_n))) - \\
2\psi\Big(\frac{d(g(x), g(w_n)) + d(g(y), g(z_n))}{2}\Big). \qquad (3.3)
\end{aligned}$$

We use the properties of ϕ and ψ, we get

$$\phi(d(g(x), g(w_{n+1})) + d(g(y), g(z_{n+1}))) \leq \phi(d(g(x), g(w_n)) + d(g(y), g(z_n))).$$

Since ϕ is nondecreasing,

$$d(g(x), g(w_{n+1})) + d(g(y), g(z_{n+1})) \leq d(g(x), g(w_n)) + d(g(y), g(z_n)).$$

Hence, $\{d(g(x), g(w_n)) + d(g(y), g(z_n))\}$ is a nonincreasing sequence, and so there exists $\ell \geq 0$ such that

$$\lim_{n\to\infty} [d(g(x), g(w_n)) + d(g(y), g(z_n))] = \ell.$$

We suppose that $\ell > 0$, and take the limit when $n \to \infty$ in (3.3), we obtain

$$\phi(\ell) \leq \phi(\ell) - 2 \lim_{n\to\infty} \psi\Big(\frac{d(g(x), g(w_n)) + d(g(y), g(z_n))}{2}\Big) < \phi(\ell),$$

a contradiction. Thus, $\ell = 0$. Now,

$$\lim_{n\to\infty} [d(g(x), g(w_n)) + d(g(y), g(z_n))] = 0,$$

i.e.,

$$\lim_{n\to\infty} d(g(x), g(w_n)) = 0 \text{ and } \lim_{n\to\infty} d(g(y), g(z_n)) = 0.$$

That is

$$\lim_{n\to\infty} g(w_n) = g(x) \text{ and } \lim_{n\to\infty} g(z_n) = g(y).$$

By using the same way, we can deduce that

$$\lim_{n\to\infty} g(w_n) = g(u) \text{ and } \lim_{n\to\infty} g(z_n) = g(v).$$

From the uniqueness of the limit, we obtain

$$g(x) = g(u) \text{ and } g(y) = g(v).$$

As g is increasing, we have

$$x = u \text{ and } y = v.$$

□

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References

Alotaibi, A. and Alsulami, S. M. (2011). Coupled coincidence points for monotone operators in partially ordered metric spaces. *Fixed Point Theory and Applications*, doi:10.1186/ 1687-1812-2011-44.(44).

Bhaskar, T. G. and Lakshmikantham, V. (2006). Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Analysis*, 65:1379–1393.

Choudhury, B. S. and Kundu, A. (2010). A coupled coincidence point result in partially ordered metric spaces for compatible mappings. *Nonlinear Analysis*, 73:2524–2531.

Choudhury, B. S. and Kundu, A. (2012). (ψ, α, β)-weak contractions in partially ordered metric spaces. *Applied Mathematics Letters*.

Lakshmikantham, V. and Ćirić, L. (2009). Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Analysis*, 70:4341–4349.

Luong, N. V. and Thuan, N. X. (2011). Coupled fixed points in partially ordered metric spaces and application. *Nonlinear Analysis*, 74:983–992.

Razani, A. and Parvaneh, V. (2012). Coupled coincidence point results for (ψ, α, β)-weak contractions in partially ordered metric spaces. *J. Appl. Math.*

Samet, B., K. E. A. H. and Cojbasic, V. (2013). Discussion on some coupled fixed point theorems. *Fixed Point Theory and Applications*, doi:10.1186/1687-1812-2013-50.